GRADIVA

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

Towards a Diagrammatic-Geometric Representation of the Notion of Speed at the Address of Primary Pupils

Aazim Jalila Achouaq [1], Bahra Mohammed [2] & Najoua Aouatif [3]

1. Thermodynamic and Energetic Laboratory, Faculty of Sciences, Languages, Communication and Education Sciences Department ENSAM, University Mohamed V, Rabat, Morocco. Email: j.aazim@um5r.ac.ma
2. Observation and Research Unit in Science and Technology, Teaching. CERMEF, Casablanca, Morocco. Email: mohamed.bahra@gmail.com
3. Laboratory of Signals, Distributed Systems and Artificial Intelligence ENSET, University Hassan II, Mohammedia /Casablanca, Morocco. Email:aouatif.najoua@gmail.com

Abstract

A straight segment, arrow (oriented) E→F, given, we present, here, a deployment of this 'arrow' giving rise to a configuration, which we describe as diagrammatico-geometric, the tracing of this segment with a vector speed $\lambda\overrightarrow{EF}$. We show that this configuration is teachable to 12-13 year olds, as an implicit model of tracing, for a given duration a priori, of a line segment. We reported this same model to students in high-grade mathematics and physics. Although this model is an update of the parametric equation of a line segment, these students are, for the most part, unable to find the elements of this update. Traditional learning objects, including the right-hand segment, landmark, location and speed, must be reworked and reinforced by learning the diagrammatic-geometric configurations presented.

1. PROBLEM STATEMENT

Coordinate systems and coordinate identification must be distinguished. Indeed, with a plane coordinate system, one can name the plane's points using the one-to-one relationship between pairs of numbers and points. This correspondence enables us to differentiate among the plane's points.

As for coordinate identification, one of its methods consists of associating the updates of rectilinear motions to the different ways of making a moving point run through a line segment. Coordinate identification is, therefore, any geometric process that determines the instantaneous positions of that point. The link between coordinate identification and the notions of velocity and velocity vectors is clear: the determination of both the distance covered by a point with a uniform rectilinear motion and the travelling time are inherent to the determination of at least two instantaneous positions of the point. This determination cannot be achieved without coordinate identification.

Yet, as soon as the notion of coordinate systems is introduced in primary school, it is separated from coordinate identification and seen as a geometric process that determines the instantaneous positions of a moving point running through a line segment. Understandably, this separation also concerns the notions of velocity and velocity vectors.

If we showed that the notion of coordinate systems is inherently linked to coordinate identification, velocity and velocity vectors it cannot make sense when it is separated from these three notions.

We would have shown that the notions of coordinate identification, velocity and velocity vectors are, as far as coordinate systems are concerned, ignored by the current didactic systems. According to G. Brousseau [1, 2], this omission is a strong didactic barrier, in school

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

institutions, to the birth of a consideration of coordinate systems that relies on the tight link between these four notions: coordinate systems, coordinate identification, velocity and velocity vectors. One can notice that this link must end up being part of school programs, as, in high school, the notion of parametric equations of line segments is studied. Indeed, as highlighted in the following paragraphs, the second notion is nothing but another version of the first one.

When seen in primary school, the connection between coordinate systems, coordinate identification, velocity and velocity vectors is the reason why the notion of velocity must be linked to coordinate identification. This link must be established through a diagrammatic-geometric representation of velocity [3]. Yet, when dealing with this notion, school practices exclusively rely on the cognitive field of arithmetic.

All this leads to the following conclusion: as the notion of velocity is not dealt with using a diagrammatic-geometric representation, school practices do not have any other choice than replacing the notions of coordinate systems, coordinate identification, velocity and velocity vectors with seemingly simple images and misleadingly intelligible calculations.

Our main argument in favour of this conclusion is the dissociation, preceded by school practices, between the notion of line segments' parametric equations and the notions of velocity and velocity vectors. This work aims at confirming this dissociation and describing its consequences on the on the notion's perspective.

2. THEORETICAL FRAMEWORK

2.1 General theoretical framework

2.1.1 The diagrammatic representation as a tool of calculations' semiotization

The definition given by P.A. Brandit [3] to the notion of diagram, considered as the substratum of the graphical thinking, extends to the diagrammatic-geometric figuration. It establishes the appropriate theoretical framework to determine the semiotization level of the objects, the images and the figures used as educational tools by school practices. Indeed, according to this author:

> "A diagram is a graphical construction which usually appears in a verbal context, and aims at representing the functioning of a phenomenon with a given causal complexity. This context can be, inter alia, a technical, theoretical or analytical discourse. It is often an epistemic discourse, as the diagrammatic representation depicts an underlying meaning, a structural or causal functioning of the phenomenon instead of its appearance. 'image of the meaning', a diagram is neither a constellation of operational, arbitrary and conventional symbols such as an equation, nor an iconic construction referring to a possible perception, such as a photograph. It is neither a written composition, nor a drawing but it is a… diagram. The diagram has oddly been forgotten in the semiological literature, which, however constantly uses it in its own graphical « illustrations ». The diagram's semiotic specificity has never been directly mentioned, despite the renowned Peirce logical diagrams". *(Translation)* [3].

Therefore, an image, an object or a picture must have some specific properties in order to become an 'image of the meaning'. For instance, a photograph of a car on the road or a flying bird is not sufficient to represent the following equation of velocity $v = \frac{d}{t}$ .The author evokes the properties that the diagrammatic representation must have:

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

"… First, there is an iconic functioning, which completes what we see or hear […]. Then, there is the construction of a causal horizon in the time dimension of an iconically established space, which enables to give a more profound meaning and a prospect of dynamical justification to what we « see », and that more or less immediately. For the time being, communication is not part of this. To enable communication, these constructions must be signified, « communicated » by a subject to another one". (Translation) [4].

The teaching of the notion of velocity in primary school must not rely on the learning of arithmetical- if not algebraic- calculations using the formula $v = \frac{d}{t}$, even when the latter is correlated to road traffic. The construction of the causal horizon mentioned in the quote must be part of the learning path concerning velocity, and this construction can only be geometrical. Moreover, this construction must be made in *the time dimension of an iconically established space*. Only in this dimension can the notions related to the motion of a moving vehicle's point acquire real meaning, as they gain *a prospect of dynamical justification* [4].

According to the author's conception of a diagram, the idea of velocity must be the mental representation of a dynamical situation. It has to be a causal construction based upon an observation made in an attention field. In addition, the attention should be focused on « *the most intriguing, ambiguous, enigmatic and heterogeneous part of an already iconically constructed field* ». [4].

Even though it can report on an *iconically* established space, a series of photographs of a moving car cannot be considered a causal horizon. One can establish a thematic segmentation thanks to the successive positions of the car, which gives this horizon the appropriate time dimension to study the notion of velocity. However, the calculations made with the formula $v = \frac{d}{t}$ which is related to this segmentation cannot lead to the "*prospect of dynamical justification"* [4]. Mentioned in the quote. Therefore, the semiotization level of these calculations drops from medium to low due to this impossibility.

The low semiotization level of these calculations leads to the following consequence: although it is *intriguing, ambiguous, enigmatic and heterogeneous* [4], the plotting's modelling of a line segment in a field that is already iconically constructed with a quantified and established speed does not seem accessible in any academic level.

2.1.2 Dissociation, through its parametric equation, of the straight line segment into a pair of straight segments in light of the epistemological recommendations from the book 'Calculations and Forms of Mathematical Activity'

In this work, we focus on the teaching phenomenon that revolves around the ambiguity in which the relationship of school didactic practices with the notion of a straight line is situated: the straight line is essentially a form that generates geometric properties of incidence. However, when plotted in a plane equipped with an orthogonal coordinate system, it can either represent the trajectory of a point moving in a straight line, the Cartesian graph of the instantaneous positions of the ordinate of this point, or the Cartesian graph of the instantaneous positions of the abscissa of this point. The parametric equation of a line combines it with these two graphs and thus provides, as an interpretation, the mentioned trajectory. This triplet (the line and the two graphs) and the modeling of uniform rectilinear movements that it represents is a blind spot in the successive didactic contracts that educational systems establish around the

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

reference frame and the line [6]. If, paraphrasing É. Benveniste [7] regarding the form of a linguistic unit, we consider that the form of a geometric object is its ability to dissociate into simpler elements, then the aforementioned triplet is part of the form of the mathematical object 'line'. This form is obscured by the didactic management of calculations accompanying the notion of a Cartesian equation. We thought it appropriate to consider that the book 'Shapes and Calculations' might be part of the theoretical framework of this work. For this reason, it is important to quote the authors of this book regarding the 'Shapes/Calculations' articulation and the stipulations it requires for teaching mathematics in general and the teaching of the notion of speed in particular at the level of compulsory education.

If we are granted that the calculations allowed by the analytic geometry of Descartes indeed relate to algebra, or even to syntax, then citing Marie Françoise Roy [8] is quite appropriate when she writes, in the conclusion of her work:

> "... From both a qualitative and a quantitative point of view, the form of the geometric object and the syntax used to describe it are intimately linked." *(Translation)*

Indeed, it is this connection that we try to restore by making available to the teacher illustrative configurations of the ability of a straight segment (as a trajectory) to dissociate into a pair of straight segments (the two Cartesian graphs of the instantaneous positions of the coordinates of the moving point whose trajectory it is).

In this work, the morphodynamic virtualities made available by dynamic geometry software, such as GeoGebra, are used so that the student can engage in observations of rectilinear movements.

If we are granted that this has some, albeit tenuous, link with the calculative experiences highlighted in the same work by B. Chevalier [9, 10] regarding the "search for singular curves, Viro's patchwork...", we would allow ourselves to cite this author when he writes:

> "... In the experimental phase, the role of visualization cannot be ignored: formulas using various alphabets do not convey the multiple characteristics of a geometric object, except those that the mathematician, looking at his sketch, has decided to explore." *(Translation)*

The parametric equation of a straight segment, in analytic geometry, is a geometric object whose one of the characteristics is the illustration it offers of the capacity of a straight segment to dissociate into a pair of straight segments. This capability is nevertheless a blind spot of school didactic practices. These practices, by deciding to explore other things besides this capability, sacrifice the form of the straight segment for the sake of calculations.

In the same work, R. Guitart [11], writes about a calculation he introduced, the calculation of assimilations, touching on the notions and problems of the two fields of seeing and saying:

"It is not at all obvious that what is conveyed as meaning by an image and what is conveyed as meaning by a discourse can be the same thing, or even just meanings translatable into one another. For what would be thus transferable, we poorly understand the natural possibilities and systematic resources; and for what can only be lost when moving from one register to another, it will be necessary to describe the classifications, under both headings." *(Translation)*

The dissociation of a straight segment into a pair of two straight segments, via, notably, the combined notions of Cartesian equation and parametric equation of straight segments, and the drawn configuration of realization of this dissociation refers to the two registers mentioned

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

in the quote. The losses mentioned therein have their parallels in the relationship of school didactic practices with this dissociation. A praxeology of a back-and-forth between seeing and saying is thus a necessity both epistemologically and epistemically. It is such a praxeology that we wish to establish in this work.

At the end of her article, in the same work, J. Boniface [12], asks whether a difference still exists today between indeterminates and variables. This question interests us greatly because the variable, commonly noted 't' in the parametric equation of a line, refers to the 'time' variable, while the Cartesian equation recalls the indeterminate of polynomials. The author concludes:

> "... This is to say that the indeterminate has acquired an even more formal character than the variable, which remains linked to a domain of values. This formal character, far from opposing the calculative and algorithmic aspect of mathematics, is instead closely intertwined with it. The polynomial thus becomes again what the algebraic (or analytic) expression was for mathematicians of the seventeenth and eighteenth centuries: a finite set of operations. And mathematics appears as what they are: less an ontology than a praxeology." *(Translation)*

The non-summoning of the time variable and the non-summoning of the modeling of uniform rectilinear motion by school didactic practices when they address the notions of rectangular coordinates, of a line, and of a line equation, show that for these practices mathematics is rather an ontology than a praxeology.

In the same work, M. Bourdeau [13] discusses the notion of informal rigor, drawing on the works of Georg Kreisel and the fight he waged against Hilbertian formalism. Let us quote Bourdeau on the matter:

> "... It is indeed traditional to associate rigor with form. As for the foundations of mathematics, the best, if not the only way to proceed is formally. Moreover, do we not say of a logical reasoning that it concludes vi formae? In this way, the formalists have appropriated to their benefit the idea of rigor. The intuitive notions being unreliable, there is no salvation but in formalism. To which Kreisel objected that this was a tenacious but unfounded prejudice, and moreover, dangerous in that it restrictively limited the field of admissible questions." *(Translation)*

The relationship of school didactic practices with the geometric figure indicates that this prejudice denounced by Kreisel among metamathematicians has also taken hold of these practices. Indeed, although the geometric figure is no longer what it was before the advent of dynamic geometry software, with which we are rather in the presence, not of fixed figures, but of true morphological potentialities, these practices continue to maintain a relationship of suspicion towards reasoning when it relies on a figure. The proof is the dissociation described above. E. Audureau and G. Crocco [14], dealing in the same work with intuitionism and constructivism at Brouwer, write:

> "... When the concepts of our knowledge are concepts of the understanding, constructed from the pure or empirical material of intuition through the categories, we can apply to them without any risk the logical laws (including the law of excluded middle) because the objects they determine do not exceed the limits of the possibility of experience. When, however, we move from concepts to the ideas of reason which, themselves, are not the product of this construction anchored on the knowing subject, the antinomies of pure reason threaten and the application of the excluded middle is suspended." *(Translation)*

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

The notion of average speed is taught to elementary school students. And, in this teaching, this notion is not immediately followed by the notion of instantaneous speed. However, when, from a given moment, any factor causing the variation of the speed of a mobile is inhibited then, from that moment, the instantaneous and average speeds become confused. Conceptually, the two notions are consubstantially linked. By breaking this link, school didactic practices are forced to put students in this situation where they must surreptitiously move "from concepts to the ideas of reason which, themselves, are not the product of a construction anchored on the knowing subject...." and this, with the consequences noted by the authors of the quote above. With the planar configurations that we propose in this work, we attempt to make available a pure and empirical material of intuition that is similar to that described in the quote and that is specific to the notion of speed, this notion conceived for the students of middle school, or even elementary school.

2.2 Specific theoretical framework: Fundamental configuration of the plotting of the line segment [EF] according to the velocity vector $\lambda\overrightarrow{\boldsymbol{EF}}$.

2.2.1 Initial elements of the fundamental configuration of the plotting of the line segment [EF] according to the velocity vector $\overrightarrow{\boldsymbol{EF}}$

The initial elements are as follows:

- The segment [EF]
- 4 lines (OU), (UI), (IV) and (VO) such that the quadrilateral OUIV is a square

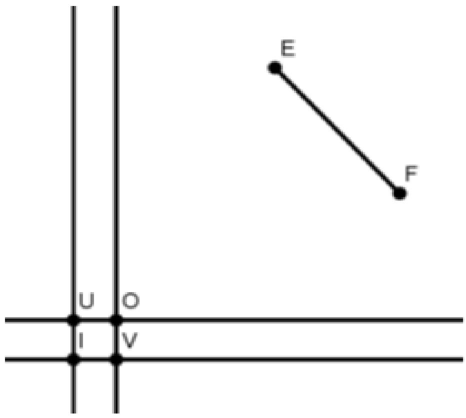

Figure 1: Initial elements of the fundamental configuration of the plotting of the line segment [EF] according to the velocity vector $\overrightarrow{\boldsymbol{EF}}$

2.2.2 Background of the configuration

- E', intersection of the parallel line to (OU) passing through E and the parallel line to (OV) passing through F
- F', intersection of the parallel line to (OV) passing through E and the parallel line to (OU) passing through F
- J, intersection of (OV) and (EE');
- J', intersection of (UI) and (EE');
- K, intersection of (UI) and (FF');
- K', intersection of (OV) with (FF');
- G, intersection of (OU) with (EF');
- G', intersection of (OU) with ((E'F);
- H, intersection of (IV) with (E'F);
- H', intersection of (IV) with (EF').

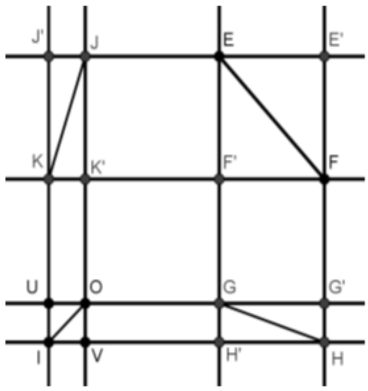

Figure 2: The configuration background

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

2.2.3 The configuration components

2.2.3.1 Determination of the position as a function of time

T, an arbitrary point of [OI]:

- T_1, intersection of [GH] with the parallel line to (OU) passing through T;
- T_2, intersection of [JK] with the parallel line to (OV) passing through T;
- P_T, the point such that $T_1 T T_2 P_T$ is a rectangle.

<u>**P_T belongs to [EF].**</u>

For every point T in [OI], there is one and only one real number t_T such that $0 \le t_T \le 1$ and $OT = t_T OI$. Yet, according to Thales' theorem : $OT = t_T OI \Leftrightarrow EP_T = t_T EF$. We consider that a moving point M runs through [EF] so that, at each instant t_T with $0 \le t_T \le 1$, the latter instant being represented by the point T, P_T is the position of that point at that instant.

t_T and P_T will be noted t and P_t

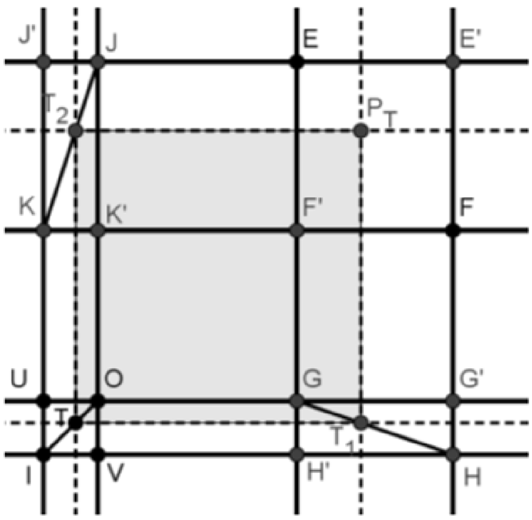

Figure 3: Determination of the position as a function of time

2.2.3.2 Determination of the instant as a function of the position

P, an arbitrary point of [EF]:

- P_1, intersection of [GH] with the parallel line to (OV) passing through P;
- P_2, intersection of [JK] with the parallel line to (OU) passing through P;
- T_P, the point such that $P_1 P P_2 T_P$ is a rectangle.

<u>**T_P belongs to [OI].**</u>

For every point P in [EF], there is one and only one real number t_p such that $0 \le t_p \le 1$ and $EP = t_p EF$. Yet, according to Thales' theorem: $EP = t_p EF \Leftrightarrow OT_p = t_p OI$. We consider that a moving point M runs through [EF] so that, each position P of this point is associated with an instant t_p with $0 \le t_p \le 1$, where M occupies that position. The instant t_p is represented with the point T_P.

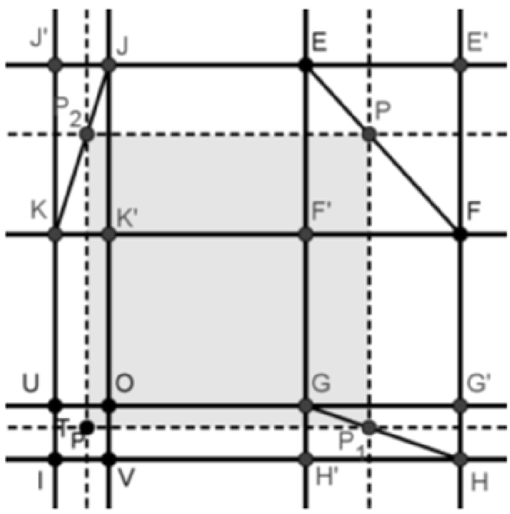

Figure 4: Determination of the instant as a function of the position

GRADIVA

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

2.2.4 Fundamental configuration of the plotting of the line segment [EF] according to the velocity vector $\overrightarrow{EF}$

One notes that P_t, P_T and that t, t_T, so one has $\bigcup_{0\le t\le 1}\{P_t\} = [EF]$: The point M runs through $[EF]$ in one unit of time, which means that the plotting of $[EF]$ is made with a speed of EF unit of length per unit of time. In addition to the speed of the plotting, one must consider the orientation (the one of (EF)) and the direction (from E to F). Therefore, it seemed accurate to name this configuration "fundamental configuration of the plotting of the line segment [EF] according to the velocity vector $\overrightarrow{EF}$ ". Indeed, the point M goes from E to F, with a speed of EF unit of length per unit of time, and it passes through every position $(P_t)_{0\le t\le 1}$ in one second.

Important remark:

Behind the identity « $\bigcup_{0\le t\le 1}\{P_t\} = [EF]$ », lies the coordinate identification of the instantaneous positions of the moving point M.

2.2.5 Configuration of the plotting of [EF] according to the velocity vector $\lambda\overrightarrow{EF}$, with λ a real number

2.2.5.1 Independent coupling: Coupling [EF$_\lambda$] to [OI], with F$_\lambda$, a point of (EF)

Let E and F denote two distinct points of the plane. Let F_S, be the symmetrical point of F relative to E. Let F' stand for a point of (EF). When F' is on the half-line [EF), $\overrightarrow{EF'} = \frac{EF'}{EF}\overrightarrow{EF}$ and when F' is on the half-line [EF$_S$), $\overrightarrow{EF'} = -\frac{EF'}{EF}\overrightarrow{EF}$ Therefore, to every real number λ corresponds one and only one point F of (EF) such that $\overrightarrow{EF}_\lambda = \lambda\overrightarrow{EF}$.

Let λ be a real number, and F$_\lambda$ the matching point. Let's consider the fundamental configuration of the plotting of [EF$_\lambda$] according to the velocity vector $\overrightarrow{EF_\lambda}$.

As a reminder, this configuration is composed of the square OUIV and the rectangles EE'F$_\lambda$F'$_\lambda$, GG'$_\lambda$H$_\lambda$H'$_\lambda$ and JJ'$_\lambda$K$_\lambda$K'$_\lambda$, as shown in the following figure:

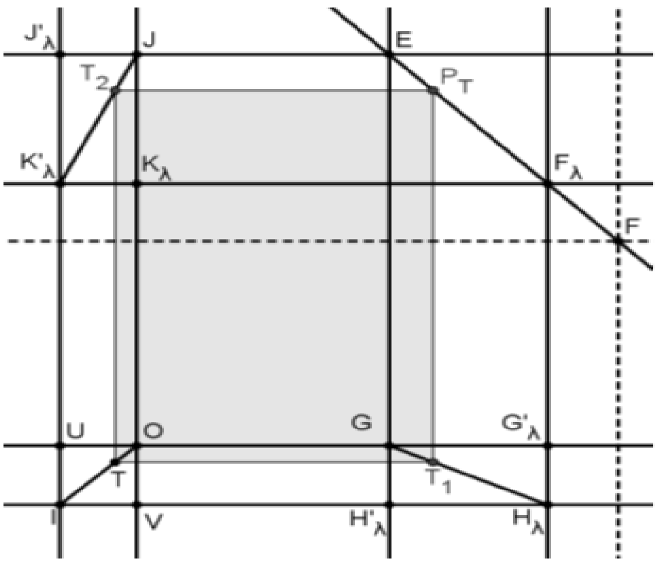

Figure 5: Coupling of [EF$_\lambda$] and [OI], with F$_\lambda$, a point of [EF]

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

2.2.5.2 Dependant coupling: Coupling of [EF] and [OI$_\lambda$], with I$_\lambda$ a point of (OI) that depends on F$_\lambda$.

The configuration of the plotting of the segment [EF] according to the velocity vector $\lambda\overrightarrow{EF}$ can be derived from the precedent one. Indeed, let's consider the following:

- H (respectively G'), the intersection of (GH$_\lambda$) (respectively (OU)) and the parallel line to (OV) passing through F;
- K' (respectively K), the intersection of (J K'$_\lambda$) (respectively (OV)) and the parallel line to (OU) passing through F;
- I'$_\lambda$ (respectively, V$_\lambda$, H'), the intersection of (OI) (respectively (OV), (EF')) and the parallel line to (OU) passing through H;
- I$_\lambda$ (respectively, U$_\lambda$, J'), the intersection of (OI) (respectively (OU), (EE')) and the parallel line to (OV) passing through K'.

The configuration below is obtained as follows:

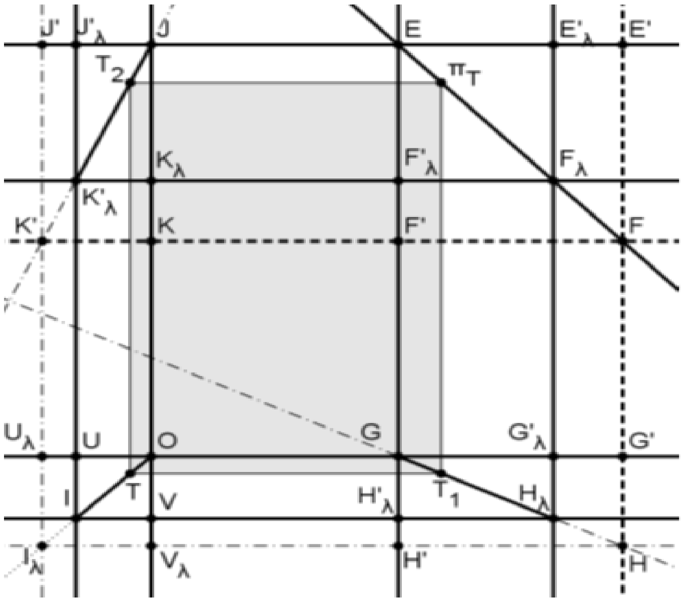

Figure 6: Coupling of [EF] and [OI$_\lambda$], with I$_\lambda$, a point of (OI) that depends on F$_\lambda$.

The properties below can be deduced as follows:

$\overrightarrow{EF_\lambda} = \lambda\overrightarrow{EF}$ so, likewise, $\overrightarrow{GH_\lambda} = \lambda\overrightarrow{GH}$, $\overrightarrow{JK'_\lambda} = \lambda\overrightarrow{JK'}$, $\overrightarrow{OI} = \lambda\overrightarrow{OI_\lambda}$ and $\overrightarrow{OI} = \lambda\overrightarrow{OI'_\lambda}$, so $I_\lambda \equiv I'_\lambda$ and $\overrightarrow{OI_\lambda} = \frac{1}{\lambda}\overrightarrow{OI}$ for $\lambda \neq 0$.

In what follows, we shall consider $\boldsymbol{\lambda \neq 0}$. From the identity $\overrightarrow{OI_\lambda} = \frac{1}{\lambda}\overrightarrow{OI}$, we can deduce,

knowing that for every point T$_\lambda$ in [OI$_\lambda$], there is one and only one real number t$_\lambda$ such that $0 \leq t_\lambda \leq 1$and $\overrightarrow{OT_\lambda} = t_\lambda\overrightarrow{OI_\lambda}$, that $\overrightarrow{OT_\lambda} = t_\lambda\frac{1}{\lambda}\overrightarrow{OI}$. Let $t = t_\lambda\frac{1}{\lambda}$, so that $0 \leq t \leq \frac{1}{\lambda}$.

In the new configuration, with the same construction of Π_t for every T$_\lambda$ in [OI$_\lambda$], than the one of P$_t$ for every T in [OI], F$_S$ being the symmetrical point of F relative to E, we obtain:

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

$$\begin{cases} \text{for } \lambda > 0, \bigcup_{0\le t\le \frac{1}{\lambda}}\{\Pi_t\} = [EF] \\ \text{for } \lambda < 0, \bigcup_{0\le t\le \frac{1}{\lambda}}\{\Pi_t\} = [EF_s] \end{cases}$$

Therefore, the points Π_t, with $0 \le t \le \frac{1}{\lambda}$, are the instantaneous positions of a moving point M running through (EF). If $\lambda > 0$, M runs through [EF], from E to F in $\frac{1}{\lambda}$ unit of time and if $\lambda < 0$, M runs through [EF_S] from E to F_S in $-\frac{1}{\lambda}$ unit of time.

The new configuration is about making a moving point M run through [EF] (respectively [EF_S]) in $\frac{1}{\lambda}$ unit of time (respectively -$\frac{1}{\lambda}$ unit of time). In both cases, the speed is $|\lambda|EF$ unit of length per unit of time. We decide to name this new configuration: ***«Fundamental configuration of the plotting of [EF] according to the velocity vector $\lambda\overrightarrow{EF}$ ».***

Important remark: One can note that the identities $\begin{cases} \text{pour } \lambda > 0, \bigcup_{0\le t\le \frac{1}{\lambda}}\{\Pi_t\} = [EF] \\ \text{pour } \lambda < 0, \bigcup_{0\le t\le \frac{1}{\lambda}}\{\Pi_t\} = [EF_s] \end{cases}$ formalize the coordinate identification of the instantaneous positions of the moving point M.

2.2.6 Junction of several configurations for the plotting of a constellation of line segments

2.2.6.1 Initial elements for the configuration of the successive plottings of two line segments

For the plotting of [EF], immediately followed by the plotting of [E'F']:

- the initial elements of the fundamental configuration of the plotting of [EF] are, apart from the segment [EF], the square OUIV;
- the initial elements of the fundamental configuration of the plotting of [E'F'] are, apart from the segment [E'F'], the square IU'I'V', which results from the $\vec{OI}$ vector translation of the square OUIV.

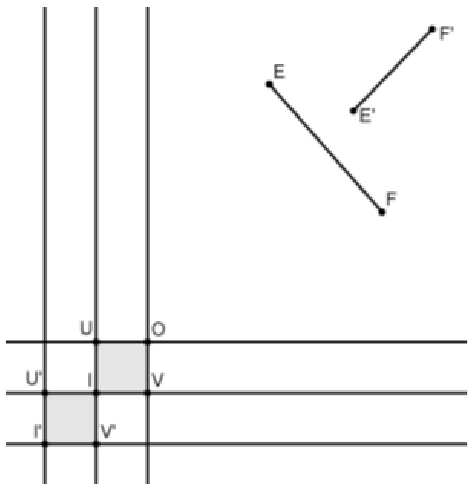

Figure 7: Initial elements of the plotting of [EF] according to the velocity vector $\overrightarrow{EF}$, followed by the plotting of [E'F'] according to the velocity vector $\overrightarrow{E'F'}$.

2.2.6.2 From the fundamental configuration of the plotting of a line segment to the encoding, in a grid pattern, of the plotting of constellations of line segments.

In a grid pattern, when the end points of the segments that must be plotted are on the nodes of the grid, squares like OUIV and IU'I'V' are already constructed. Rectangles (and their

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

diagonals) such as GG'HH' and JJ'KK' (seen in the fundamental configuration) are already constructed as well. These rectangles' diagonals enable the encoding of the plotting of the constellation of the line segments considered. This encoding is illustrated in the figure below.

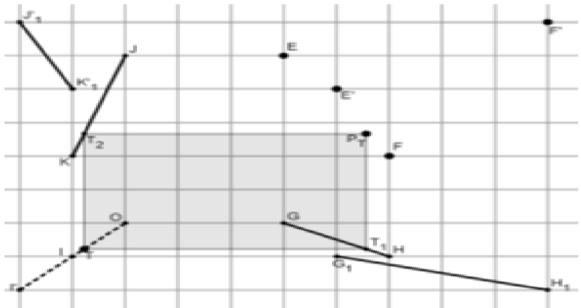

Figure 8a: Encoding of $[EF]\cup[E'F']$

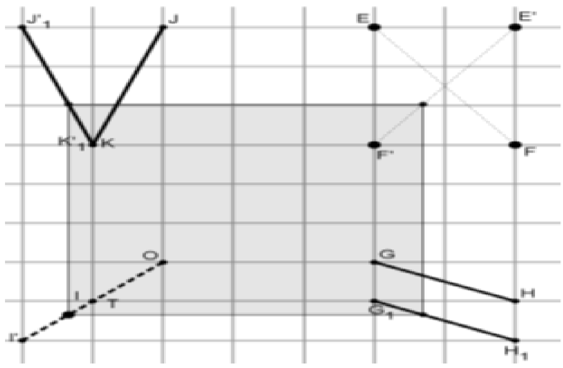

Figure 8b: Encoding of the letter X

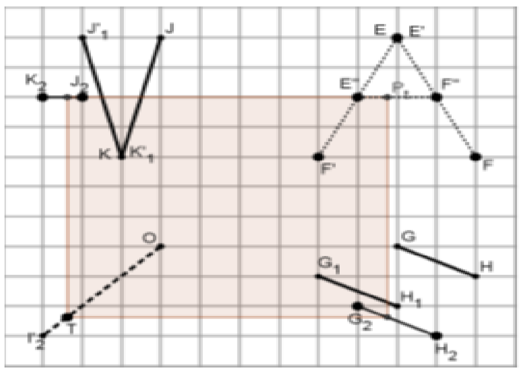

Figure 8c: Encoding of the letter A

- In the figures (8a) and (8b), the segments [OI'], [GH], [JK], [G_1H_1] and [$J'_1K'_1$]] encode the plotting of the segment [EF] followed by the plotting of the segment [E'F']. Indeed, when T runs through [OI'], the point P_T runs through [EF]∪[E'F'] ; in the figure (8b), the 5 segments encode the letter X;
- In the figure (8c), the segments [OI'], [GH], [JK], [G_1H_1], [$J'_1K'_1$] and [J_2K_2] encode the plotting of the constellation of segments [EF] ∪ [E'F'] ∪ [E''F''] (the capital letter A).

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

3. THE RESEARCH GOAL AND HYPOTHESIS

What we present as "*Fundamental configuration of the plotting of [EF] according to the velocity vector* $\lambda\overrightarrow{EF}$'' is nothing but a diagrammatic-geometric figuration of the notion of velocity. It is also a geometrical update of the notion of line segments' parametric equation.

In view of this representation of the first notion and this update of the second one, this research aims at specifying the perspective of school institution on the second one. One can note that the second one subsumes the first one, and it could not have a simpler geometrical update.

3.1 The research goal

3.1.1 The approach to line parametric equations that should be promoted in secondary school

Considering the nature of the mathematical notion of line segments' parametric equations, the right approach to these equations should tightly link them to the notions of coordinate identification, velocity and velocity vectors[1]

However, this tight link has been forgotten by the didactic systems: parametric equations of line segments are never explicitly linked, by school practices, to the notion of coordinate identification, and the link is even weaker with the notions of velocity and velocity vectors.

3.1.2 The research goal

The research goal is to show that the approach to line segments' parametric equations that should be promoted in secondary school is the one based upon the tight link in question. This would imply that the fundamental configuration of the plotting of line segments [EF] according to the velocity vector $\lambda\overrightarrow{\mathrm{EF}}$ should be a teaching tool and a learning object as early from primary school level upwards.

This demonstration requires arguments in favour of the following conclusion: the vast majority of students would be unable to perceive, spontaneously, the tight link between line segments' parametric equations and the notions of coordinate systems, coordinate identification, velocity and velocity vectors.

This amounts to showing that, when shown a geometric construction related to the configuration of the plotting of a line segment [EF] according to the velocity vector $\lambda\overrightarrow{\mathrm{EF}}$, the vast majority of students is unable to detect a modelling of the plotting of line segments based upon an update of the notion of parametric equations.

In particular, we will show that the students are unable to decipher the encoding of constellations of line segments, done in a grid pattern, and previously described in this work. This argument will be strengthened by the fact that primary school students are able to use, in an implicit way, the fundamental configuration of the plotting of a line segment to encode, in a grid pattern, the plotting of capital letters (seen as constellations).

The letters used are: A, Z, E, T, Y, I, F, H, K, L, M, W, X, V and N. This goal is an extension of similar goals studied in other works [15-19]

3.2 The research hypothesis

We make the hypothesis according to which it is possible to make middle school students (12-13 years old) use, in an implicit way, the fundamental configuration of the plotting of line

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

segments [EF] according to the velocity vector $\overrightarrow{EF}$ in order to encode, in a grid pattern, the plotting of the following letters: A, Z, E, T, Y, I, F, H, K, L, M, W, X, V and N (these letters are seen as constellations of line segments). We also suppose that, to this end, it is sufficient to: (1) train these students with one or two examples among these letters; (2) teach them how to check the proposed encoding by using, implicitly, the identity « $\bigcup_{0\leq t\leq 1}\{P_t\} = [EF]$ » that illustrates the coordinate identification of the instantaneous positions of the moving point M, which runs through the segments that compose the considered letter.

In contrast, these encodings will not be understood by engineering school students as a modelling of the plotting of line segments based upon an update of the notion of segments' parametric equations. In most cases, these students' knowledge about parametric equations will not be of any help when trying to decipher these encodings.

4. THE RESEARCH METHODOLOGY

In order to show that the hypothesis of the research is true, we asked middle school teachers to train their students (12-13 years old) with the encoding of the letters Z, E, T, Y, I, F, H, K, L, M, W, V and N, with the letters A and X as examples. The identity « $\bigcup_{0\leq t\leq 1}\{P_t\} \equiv [EF]$ » that illustrates the coordinate identification of the instantaneous positions of the moving point M, which runs through the segments that compose the considered letter, is implicitly used to check the encoding.

The encodings made by these students are then shown to engineering school students, who must study and interpret them. The engineering school students are provided with the following information: (1) the geometric figures that they are shown are updates of a configuration that models the plotting of constellations of line segments, and they encode this plotting; (2) when the configuration deals with a line segment [EF], it models the plotting and the speed of the plotting, and this speed lies behind the vector $\overrightarrow{EF}$, seen as a velocity vector ; (3) this plotting is an update of the notion of parametric equation of a line segment.

Provided with that information, the engineering school students then have to: (1) put aside the students' productions after having reviewed them; (2) put themselves in the place of these students and make the same encodings; (3) explain how these encodings are an update of the notion of line segments' parametric equations.

Contrarily to the middle school students, the engineering school students were not trained for these encodings.

At the end of the experiment, the success rates in the constructions related to the encodings must be examined, in the cases of middle school students and engineering school students. Two options are to be considered, concerning these success rates:

- Option O1:
  - O11: more than 75% of the answers of middle school students are correct;
  - O12: less than 75% of the answers of middle school students are correct;
- Option O2:
  - O 21: more than 25 % of the answers of engineering school students are correct;
  - O 22: less than 25 % of the answers of engineering school students are correct.

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

We consider that the research hypothesis is true as long as the results obtained are O11 for option O1 and O22 for option O2. Indeed, O11 can be interpreted the following way: the fundamental configuration of the plotting of a line segment [EF] according to the velocity vector $\lambda\overrightarrow{EF}$gives an encoder of the plotting of constellations of line segments of which end points are the nodes of a grid pattern. This encoder is sufficiently manageable to suppose that 12-13 years old students can quickly master it, thanks to a training supervised by their mathematics teacher. After this training, they are able to construct the encoding of the plotting of every capital letter, seen as constellations. The manageability of the encoder is obvious if more than 75% of middle school students successfully encode the plotting of these letters, by following the instructions of the encoder. The confirmation of this obviousness made us consider option O1.

As for the consideration of option O2, it can be interpreted the following way: when, in the students' productions, the encoding of a letter is correct, the observer sees the segments that constitute a given capital letter, as well as three other groups of segments. The observer has to notice that the letter is in one of the quadrants of the grid pattern, while the three other groups of segments are located in the three remaining quadrants. As this process is a program that enables the plotting of the letter, it must be obvious for the observer that the three groups of segments are interpreted as an update of a function of which outputs set is the letter. The inputs set must, therefore, be a unique segment. This segment is located in the third quadrant. The groups of segments of the second and the fourth quadrants are only auxiliary groups for the construction of the image of the segment in the third quadrant. This construction can rely on the vertical and horizontal projections, that is to say the two projections made according to the axes that delimit the four quadrants. The observer, thanks to this function, must see the encoder as a commodification of the parametric equation of the first quadrant's segment. Therefore, the observer must master the notions of functions and line segments' parametric equations. The engineering school students can, potentially, reach the level required, but some factors can prevent that. This is the reason why we consider option O2. If a significant part of the engineering school students succeeds in the experiment, even if it is a minority, we will be obliged to consider that the fundamental configuration of the plotting of a line segment [EF] according to the velocity vector $\overrightarrow{\mathrm{EF}}$ was, overall, understood by the engineering school students.

In fact, the two options do not enable us to conclude on the potential or real abilities of the middle school and engineering school students. They only help determine whether the fundamental configuration of the plotting of a line segment [EF] according to the velocity vector $\lambda\overrightarrow{\mathrm{EF}}$ can be used to make the notions of coordinate systems, coordinate identification, velocity and velocity vectors intelligible, and whether this configuration is an interesting update of the notion of line segments' parametric equations.

5. CHARACTERISTICS OF THE TESTED POPULATION

33 middle school students, aged 12 or 13, took part to the experiment. They study in 2nd year of middle school (school year 2015-2016, HIBA School, city of Rabat, MOROCCO)

141 engineering school students took part in the experiment. Some of them study in 2nd year in the ENSET of Rabat (Rabat, MOROCCO, school year 2022-2023, branches: Electrical Engineering, Biomedical Engineering, Industrial Conception and Production); others in 2nd year in the ENSET of Mohammedia (Mohammedia, MOROCCO, school year 2022-2023, branches: Industrial Shipping Management, Electrical Systems and Renewable Energies,

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

Logistic Engineering and Distributed Information Systems, Distributed Computing System); the remaining others are in 2nd year in the Mohammedia engineering school (Rabat, MOROCCO, school year 2022-2023, branch: Computer Engineering).

6. RESULTS AND ANALYSIS OF THE EXPERIMENT

The engineering school students need a lot more assistance to see that the problem lies in the deciphering of the encoding of the capital letters made by the middle-school students. Among the 141 future engineers concerned, 80 sent an answer and only 6 answers were correct, that is to say less than 1% of the answers. These results do not put into question these student's mastery of the mathematics of their level. They actually show that their knowledge about the notion of line segments parametric equations does not enable them to comprehend the encodings and decipher them. For these students, their mastery of this notion being guaranteed, any questioning that is not related to this mastery – about this notion's semantisation and intelligibility for instance- is out of ordinary.

These student's approach to mathematical objects and notions, such as parametric equations, coordinate systems, coordinate identification, velocity and velocity vectors is more about the syntactic validity of calculations than the semantic validation of the enunciation.

The middle school students' results show that they were able to take advantage of the training they had, in order to encode the plotting of line segments using, in an implicit way, the fundamental configuration of the plotting of the line segments [EF] according to the vector $\overrightarrow{\mathrm{EF}}$. 66 answers were received, and among these answers, 62 were correct, that is to say more than 90% of the answers. These results show that the configuration is, as an encoder of the plotting of line segments, sufficiently manageable to be a full-fledged learning object. Therefore, didactic systems can rely on this encoder in order to take into account the notions of coordinate identification, velocity and velocity vectors when dealing with coordinate systems.

The low results obtained by the engineering school students constitute a strong argument in favour of this incorporation.

7. CONCLUSION

This research was about the way the notions of coordinate systems, coordinate identification, velocity and velocity vectors are dealt with by the current school practices. We first showed that these four notions are tightly linked and that the notion of line segments' parametric equations lies behind this tight link. We gave this notion a diagrammatic-geometric interpretation, which gave birth to what is called here: *«Fundamental configuration of the plotting of line segments [EF] according to the velocity vector* $\lambda\overrightarrow{\mathrm{EF}}$ *».*

This configuration shows that the tight link between these four notions is a latent complexity in this equation. It also shows that school practices tend to consider that the taught knowledge related to this equation enables the students to spontaneously and conveniently use this complexity. It seems that, however, the tight link in question has crystallized so deeply that it is really unlikely to see this complexity being used without former explicit teaching.

Therefore, while this configuration is hermetically sealed in the face of engineering school students, it is easily manageable for middle school students, which means that it could be part of the learning objects starting from primary school.

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

The dissociation, proceeded by school practices in high-school and even beyond, between the notion of line segments' parametric equations and the notions of velocity and velocity vectors, has great consequences on the perspective of the school institution on the notions of coordinate systems, coordinate identification, velocity and velocity vectors. Due to this dissociation, these three last notions have been disconnected from the one of coordinate systems by didactic systems.

Foot Notes

1) Indeed, let a, b, c, d be 4 real numbers. The plane being associated to a direct orthonormal system $(O,\vec{\imath},\vec{\jmath})$, the union $\bigcup_{t\in IR}\{P_t\}$, with $\begin{cases} x_{Pt} = at + b \\ y_{Pt} = ct + d \\ \quad t \in IR \end{cases}$ (S) is the set of all possible instantaneous positions for the point M. It also formalizes, algebraically, the process of coordinate identification of the point M. This point has a rectilinear motion, as the system (S) is the parametric representation of the line which has $\vec{u}\binom{a}{c}$ as a direction vector. This motion is uniform, as for every $(t,\lambda)\in {}^2$, $\overrightarrow{P_tP_{t+\lambda}}\binom{-\lambda a}{-\lambda c}$, does not depend on t : this shows that the distance between two given positions P_t and $P_{t+\lambda}$ is proportional to the time $|\lambda|$ that the point M needed in order to go from P_t to $P_{t+\square}$. The proportionality coefficient is $\sqrt{a^2+c^2}$. Therefore, the velocity of M is $\sqrt{a^2+c^2}$ unit of length per unit of time. This can all be summarised in the following sentence: *«M has a rectilinear uniform motion of which velocity vector is* $\overrightarrow{P_tP_{t+1}}$*»*.

References

1) Brousseau, G. (1998). Les obstacles épistémologiques, problèmes et ingénierie didactique. La théorie des situations didactiques, 115-160.
2) Brousseau, G. (2010). Class note. Références épistémologiques pour la didactique des mathématiques [PowerPoint slides]. http://guy-brousseau.com/1489/cours-2010-2011-diaporama-additionnelreferences-epistemologiques-pour-la-idactique-des-mathematiques/
3) Brandt, P. A., (2012). La pensée graphique - Pour une sémiotique des diagrammes. Visible, 9. Visualisation et mathématisation. ISBN : 2-84287-575-6. https://www.academia.edu/5066775/Pour une s´sémiotique des diagrammes 2012
4) Brandt, P. A., (2014). Sens et modalités - dans la perspective d'une sémiotique cognitive. Actes Sémiotiques. 117. ISSN : 2270-4957. http://epublications.unilim.fr/revues/as/pdf/5085
5) Boniface, J. (2003). "Calculs et Formes de l'activité mathématique", Ed. Ellipse, ISBN 2-7298-1586-4.
6) Mercier, A. (1999). "Le contrat didactique dans la théorie des situations didactiques en mathématiques". Premières Journées Québécoises de Didactique des Mathématiques, hosted by Gisèle Lemoyne, Université de Montréal, Montréal, Canada. hal-01991644.
7) Benveniste E. (1974), "Problèmes de Linguistique Générale 12". Paris : Gallimard.

Volume 63 | Issue 07 | July 2024
DOI: 10.5281/zenodo.12798816

8) Roy M.F. (2003), ”Formes et formules : les algorithmes de la géométrie Algébrique”, in ”Calculs et Formes de l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses ´Edition Marketing S.A.
9) Chevalier B. (2003), ”Perturbations des courbes algébriques réelles planes singulières de degré donné (de 1876 à 2000)”, in ”Calculs et Formes De l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses ´Edition Marketing S.A.
10) Chanudet M., (2019) Etude des pratiques évaluatives des enseignants dans le cadre d’un enseignement centré sur la résolution de problèmes en mathématiques. Switzerland : Université de Genève ; 2019
11) Guitart R., (2003), ”Calculs d’assimilations, modalités et analyse d’images”, in ”Calculs et Formes de l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses Edition Marketing S.A.
12) Boniface J. (2003), ”De l’indéterminée à la variable”, in ”Calculs et Formes De l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses ´Edition Marketing S.A.
13) Bourdeau M. (2003), ”La rigueur informelle”, in ”Calculs et Formes De l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses ´Edition Marketing S.A.
14) Audureau E. and Crocco G., (2003), ”Intuitionnisme et constructivisme chez Brouwer”, in ”Calculs et Formes De l’activité mathématiques”, ISBN 2-7298-1586-4 Ellipses ´Edition Marketing S.A.
15) El Abbassi F., Bouanani H. (2013) “La qualité des systèmes d’enseignement des sciences et techniques à l’aune des fondements épistémologiques des sciences cognitives ” The journal of quality in education N° 5, PP 99 -132, Legal Deposit : 2011, Issn : 2028-1897.
16) El Abbassi F., Bouanani H., (2014), Défaut de traçabilité des objets de savoir et défaut d’objets d’apprentissage In Educational Journal of the University of Patras UNESCO Chair 2014, 1(1), ISSN : 2241-9152.
17) Jawad E., (2015), La nécessité de la mobilisation des variables didactiques spatiotemporelles dès les premiers enseignements de la géométrie In Educational Journal of the University of Patras UNESCO Chair 2014, 1(1), ISSN : 2241-9152.
18) El Abbassi F, Bouanani H., (2015), Quelles pratiques didactiques à propos des formules trigonométriques à l’aune de la généralisation des TICE ? In Educational Journal of the University of Patras UNESCO Chair 2014, 2(1), ISSN: 2241-9152.
19) Talbi, M. (2017). The Inclined Plane, Simple Machine and Calculus Geometric Register. China-USA Business Review, 16, 31-40. doi: 10.17265/1537-1514/2017.01.004